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In the framework of inflationary cosmology I study some aspects of nonequilibrium thermodynamics for the matter field fluctuations. The thermodynamic analysis is developed for de Sitter and power - law expansions of the universe. In both cases, I find that the heat capacity is negative leading respectively, to exponential and superexponential growth for the number of states in the infrared sector for de Sitter and power-law expansions of the universe. The spectrum for the matter field fluctuations can be understood from the background effective temperature when the horizon entry.

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Since inflation stretches microscopic scales into astronomical ones, it suggests that the density perturbations which provide the seeds for galaxy formation might have originated as microscopic quantum fluctuations [1,2].

A promising approach towards a better understanding of these phenomena is the paradigm of stochastic inflation. The most widely accepted approach assume that the inflationary phase is driving by a quantum scalar field φ with a potential $V(\varphi)$. Within this perspective, the stochastic inflation proposes to describe the dynamics of this quantum field on the basis of a splitting of φ in a homogeneous and an inhomogeneous components. Usually the homogeneous one is interpreted as a classical field that arises from a coarse-grained average over a volume larger than the horizon volume, and plays the role of a global order parameter [3]. All information on scales smaller than this volume, such as the density fluctuations, is contained in the inhomogeneous component. During inflation vacuum fluctuations on scales less than the Hubble radius are magnified into classical perturbations in the scalar fields on scales larger than the Hubble radius. The primordial perturbations arise solely from the zero - point fluctuations of the quantized fields. Although the region which ultimately expanded to become the observed universe may have contained excitations above the vacuum, these excitations would not have any significant effect on the present state of the universe because a sufficiently large amount of the inflation would have redshifted these excitations to immeasurably long wavelengths. The zero - point fluctuations, on the other hand, have arbitrarily small wavelengths, indeed they are most significant at very small length scales. The zero - point fluctuations are imperceptible because of their short wavelengths, but the process of inflation can strech these wavelengths to macroscopic and eventually to astronomical dimensions. Hence, the density perturbations should be responsible for the large scale structure formation in the universe.

In this paper I am interested in the study of the termodynamical properties of matter field fluctuations in the infrared (IR) sector. Some thermodynamical researchs were developed recently in the framework of the density topology of the spacetime foam [4] and the quantum structure of Schwarschild black holes [5]. During inflation this sector varies the number of degrees of freedom. This is an unstable sector which describes the universe on a scale much larger than the observable universe. A natural consequence of this approach is the self - reproduction of universes and the return to a global stationary picture. For inflation the simplest assumption is that there are two scales: a long - time, long - scale associated with the vacuum energy dynamics, and the short - time, short - distance scale associated with a random force component. The Hubble time $1/H$, separates the two regimes.

The dynamics of a scalar field minimally coupled to a classical gravitational one is described by the Lagrangian

$$\mathcal{L}(\varphi, \varphi_{,\mu}) = -\sqrt{-g}\left[\frac{R}{16\pi} + \frac{1}{2} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + V(\varphi)\right], \quad (1)$$

where R is the scalar curvature $g^{\mu\nu}$ are the components of the metric tensor (with $\mu, \nu = 0, 1, 2, 3$).

The equation that describes the fluctuations during the inflationary phase is [6,7]

$$\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H_c\dot{\phi} + V''(\phi_c)\ \phi = 0, \quad (2)$$

where $V''(\phi_c) \equiv \left.\frac{\partial^2 V(\varphi)}{\partial\varphi^2}\right|_{\phi_c}$. Here, the semiclassical approach $\varphi(\vec{x}, t) = \phi_c(t) + \phi(\vec{x}, t)$ was taken into account to describe the quantum fluctuations $\phi(\vec{x}, t)$, with $< 0|\varphi|0 >= \phi_c(t)$ and $< 0|\phi|0 >= 0$. Here, $|0>$ denotes the vacuum state. Furthermore, $H_c(\phi_c) = \frac{\dot{a}}{a}$ is the Hubble parameter and a is the scale factor of the universe. During inflation, the universe is accelerated $\ddot{a} > 0$ and the inflation ends when $\ddot{a} \sim 0$. The equation (2) describes the matter field fluctuations (up to nonlinear terms in ϕ), on a globally homogeneous and isotropic background spacetime described by a Friedmann - Robertson - Walker metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2. \quad (3)$$

The eq. (2) can be simplified by means of the map $\phi = e^{-3/2\int H_c dt}\chi$

$$\ddot{\chi} - a^{-2}\left[k_0^2(t) - k^2\right]\chi = 0, \quad (4)$$

where

$$k_0^2(t) = a^2\left[\frac{9}{4}H_c^2 + \frac{3}{2}\dot{H}_c - V''[\phi_c(t)]\right] \quad (5)$$

is the time - dependent wavenumber that separates the infrared and the ultraviolet sectors, which describe the two relevants scales during inflation. Since $\dot{k}_0(t) > 0$, during inflation new and new modes enters in the infrared sector such that in this sector the frequency ω_k holds

$$\omega_k^2(t) = -a^{-2}\left[k_0^2(t) - k^2\right] < 0. \quad (6)$$

As in previous works [7,8] one can write the redefined matter field fluctuations in the infrared sector by means a Fourier expansion which selects the long wavelengths modes $\chi_k = e^{i\vec{k}.\vec{x}}\xi_k(t)$ (for $k \ll k_0$)

$$\chi_{cg} = \frac{1}{(2\pi)^{3/2}}\int d^3k\ \theta(\epsilon k_0 - k)\left[a_k\chi_k + a^\dagger\chi_k^*\right]. \quad (7)$$

Thus, the coarse - grained field χ_{cg} describes the matter field fluctuations on the infrared sector. Here, $\epsilon \ll 1$ is a dimensionless parameter and $a_k^\dagger, a_k$ are the creation and destruction operators with the algebra $[a_k^\dagger, a_{k'}^\dagger] = [a_k, a_{k'}] = 0$ and $[a_k^\dagger, a_{k'}] = i\delta(k - k')$. The quantum to classical transition of the matter field fluctuations in the infrared sector is well known [7,9–11]. The field χ_{cg}

can be considered as classical when the time dependent modes of this sector holds the condition $\left|\frac{\mathrm{Im}[\xi_k(t)]}{\mathrm{Re}[\xi_k(t)]}\right|_{IR} \ll 1$.

The problem of the increasing number of degrees of freedom for the matter field fluctuations in the infrared sector in the context of thermodynamics is not well studied. To make a thermodynamic description for the fluctuations in the infrared sector we can introduce the partition function $Z(\beta)$

$$Z(\beta) \simeq \int_0^{\epsilon k_0} \frac{d^3k}{(2\pi)^3} e^{-\beta\omega_k(t)} = \int_{\omega_{k=0}}^{\omega_{\epsilon k_0}} d\omega_k \; \rho(\omega_k) \; e^{-\beta\omega_k}, \qquad (8)$$

where β^{-1} plays the role of the background "temperature" and ω_k is the relevant frequency given by eq. (6). Furthermore, I denote the squared frequency with the cut off wavenumber ϵk_0 by $\omega^2_{\epsilon k_0} = -a^{-2}\left[k_0^2(1-\epsilon^2)\right]$. Here, ϵ is a dimensionless parameter given by $k/k_0 \ll 1$. In the framework of stochastic inflation the background is well represented by the ultraviolet sector, where $\omega^2_k > 0$. In the semiclassical limit the frequency ω_k plays the role of the energy for each mode with wavenumber k. We are interested in the infrared sector. In this sector the wavenumbers are very small with respect to k_0 ($k \ll k_0$), and the frequency ω_k is imaginary pure ($\omega_k = \pm \mathrm{i}|\omega_k|$). As we will see later, the parameter β is also imaginary pure and thus the argument of the exponential in eq. (8) remains real. The function $\rho(\omega_k)$ gives the density of states with frequency ω_k on the infrared sector

$$\rho(\omega_k) = \frac{1}{(2\pi)^3}\left|\frac{d^3k}{d\omega_k}\right| = \frac{k^2}{2\pi^2}\left|\frac{dk}{d\omega_k}\right|, \qquad (9)$$

where $\left|\frac{d^3k}{d\omega_k}\right|$ is the Jacobian of the transformation, so that

$$\left|\frac{d\omega_k}{dk}\right| = \frac{\left[k_0^2 + a^2\omega_k^2\right]^{1/2}}{a^2\;|\omega_k|}, \qquad (10)$$

where $|\omega_k| = [\omega_k\omega_k^*]^{1/2}$ and the asterisk denotes the complex conjugate. The density of states with frequency ω_k is given by

$$\rho(\omega_k) = \frac{1}{2\pi^2}\left[k_0^2 + a^2(t)\omega_k^2\right]^{1/2}|\omega_k|\; a^2(t). \qquad (11)$$

The thermodynamics for systems with exponentially growth of density of states was first considered by Hagedorn in the framework of the hadron mass spectrum in bootstrap models [12,13]. Energy added to a system can go either into increasing the energy of existing states or into creating new states. In the case of the matter field fluctuations in the infrared sector new and new states are created from the ultraviolet sector.

The "temperature" and the heat capacity are given by

$$\beta = \left|\left|\frac{\partial \ln[\rho]}{\partial\omega_k}\right|\right|_{k=\epsilon k_0}, \qquad (12)$$

$$C_V = -\,\beta^2\left[\frac{\partial^2\ln[\rho]}{\partial\omega_k^2}\right]^{-1}\Bigg|_{k=\epsilon k_0}. \qquad (13)$$

The condition that the density of states rises superexponentially is precisely that the second derivative in eq. (13) be positive, and C_V thus be negative. Systems with negative heat capacities are thermodynamically unstable. They are placed in contact with a heat bath and will experience runaway heating or cooling. If the density of states grows exponentially, an inflow of energy at the Hagedorn temperature goes entirely into producing new states, leaving the temperature constant. If the density of states grows superexponentially, the process is similar, but the production of new states is so copious that an inflow of energy actually drives the temperature down. To simplify the notation, in the following I will denote $\omega_{\epsilon k_0}$ as ω. For inflationary models one obtains in the infrared sector

$$C_V = \frac{-\mu^4\left(\omega^2\mu^2 + \mu^4 + 2\omega^4\right)}{\omega^4\left(\mu^2+\omega^2\right)^4}, \qquad (14)$$

where $\mu^2 = k_0^2/a^2$. Furthermore, the inverse of the effective temperature is

$$\beta \simeq \mp\mathrm{i}\frac{\mu^2}{|\omega|\left(\mu^2+\omega^2\right)}. \qquad (15)$$

In the case we are studying, the "thermal bath" is described by the ultraviolet sector, but it is not trully thermalized. In the framework of supercooled inflation, β^{-1} it is not a trully temperature. Thus, it is imaginary pure. Furthermore, the parameter β describes the environment of the infrared sector, here characterized by the ultraviolet sector. The quantum nature of the matter field fluctuations in the ultraviolet sector is another motivation for the parameter β to be imaginary pure.

Furthermore, the heat capacity gives information about the evolution of the infrared sector, which is an unstable sector. If $C_V > 0$, the system distributes its energy in the existent states. The inverse situation describes a system which increments very rapidly the number of states.

Thermodynamics for a de Sitter expansion: As a firsth example I will consider a scale factor $a \sim e^{H_0 t}$, where H_0 is the Hubble parameter. For a de Sitter expansion this parameter is constant. In a de Sitter expansion $\mu^2 = \nu^2 H_0^2$, where $\nu^2 = \frac{9}{4} - \frac{m^2}{H_0^2}$. The density of states is given by

$$\rho(\omega_k) \simeq \frac{1}{2\pi^2}|\omega_k|\left[\nu^2 H_0^2 + \omega_k^2\right]^{1/2}\; e^{3H_0 t}. \qquad (16)$$

From eq. (12) one obtains the inverse of the effective temperature for a de Sitter expansion

$$\beta \simeq \mp \mathrm{i} \frac{1}{\nu H_0 \epsilon^2 \sqrt{1-\epsilon^2}} \simeq \mp \frac{\mathrm{i}}{\nu H_0 \epsilon^2}. \qquad (17)$$

which is imaginary pure and does not depends on time. Furthermore, the heat capacity is [see eq. (13)]

$$C_V \simeq \frac{-\left(2+2\epsilon^4-3\epsilon^2\right)}{\left(\epsilon^2 \nu H_0\right)^4 \left(1-\epsilon^2\right)^2} \simeq -2\left(\beta\beta^*\right)^2. \qquad (18)$$

Note that the heat capacity is negative but constant. This means that, as we put energy into the infrared sector, a greater and greater proportion of it is employed in the exponential production of new states rather than in increasing the energy of already existing states.

Thermodynamics for a power - law expansion: Now we consider the case where the scale factor evolves as $a \sim t^p$. In this case the Hubble parameter is $H_c = p/t$ and the effective squared mass parameter becomes [7]

$$\mu^2(t) = t^{-2}\left[\frac{9}{4}p^2 - \frac{15}{2}p + 2\right]. \qquad (19)$$

The density of states $\rho(\omega_k)$ is

$$\rho(\omega_k) \simeq \frac{|\omega_k|}{2\pi^2}\left[K^2 t^{-2} + \omega_k^2\right] t^{3p}, \qquad (20)$$

where $K = \sqrt{\frac{9}{4}p^2 - \frac{15}{3}p + 2}$. Inflation holds when $K > 0$, i.e., for $p > 3.04$. Hence, the inverse of the effective temperature of the infrared sector is [see eq. (12)]

$$\beta \simeq \mp \mathrm{i} \frac{t}{K\epsilon^2\sqrt{1-\epsilon^2}} \simeq \mp \frac{\mathrm{i}t}{K\epsilon^2}. \qquad (21)$$

Furthermore, the heat capacity is obtained from eq. (13)

$$C_V = \frac{-\left(2+2\epsilon^4-3\epsilon^2\right)t^4}{\left(K\epsilon^2\right)^4\left(1-\epsilon^2\right)^2} \simeq -2\left(\beta\beta^*\right)^2. \qquad (22)$$

The expression (22) for C_V becomes more and more negative with time, due to the unstability of the infrared sector during inflation. As was demonstrated in a previous work [7], in a power - law expansion for the universe the inflaton potential suppresses the dispersion of the quantum fluctuations in a power - law expansion for the universe. This result coincides with the quantum field prediction and could be responsible for the very rapidly decreasing of the heat capacity C_V.

General comments: If $(\beta\beta^*)^{-1/2}$ is the zero mode temperature (or background temperature), the squared infrared matter field fluctuations when the horizon entry will be

$$\left\langle\phi_{cg}^2\right\rangle_{IR} \simeq \frac{\epsilon^6}{6\pi^2(\beta\beta^*)^{3/2}}\left[\xi_{k=0}(t)\right]^2\Big|_{t=t_*}, \qquad (23)$$

where t_* is the time when the horizon entry and $\xi_{k=0}(t)$ is the solution for the zero mode equation of motion $\ddot{\xi}_0 - \frac{\epsilon^2}{\beta\beta^*}\xi_0 = 0$, so that the amplitude for primordial power density perturbations should be a function of the background temperature. In other words, if $\mathcal{P}_{\phi_{cg}}(t_*) = |\delta_k|^2$ is the power spectrum for the matter field fluctuations, such that $\left\langle\phi_{cg}^2\right\rangle_{IR} = \int_0^{\epsilon k_0}\frac{dk}{k}\mathcal{P}_{\phi_{cg}}$, hence the density perturbations can be written as $|\delta_k| = A(t_*)\ k^n$ ($n = 3/2$), which agree with the best-fit slope of COBE data: $n = 1.2 \pm 0.3$. A best approximation could be obtained from the exact solution for the equation for the modes: $\ddot{\xi}_k + \left[k^2/a^2 - \epsilon^2/(\beta\beta^*)\right]\xi_k = 0$.

To summarize, note that in the cases here developed — de Sitter and power - law expanding universes — the heat capacity is negative. This is because the density of states in the infrared sector grows exponentially or superexponentially during inflation. Hence, the increasing rate of states in the infrared sector is more copious than the inflow of energy in this sector. Rather an increasing energy density, an increasing of $|C_V|$ (for $C_V < 0$), gives a superproduction of the number of degrees of freedom in the infrared sector. The interesting fact is that in the both cases here studied one obtains $\mu^2(\beta\beta^*) \simeq \epsilon^2$ and $C_V \simeq -2(\beta\beta^*)^2$. The main difference founded in the examples here studied is that the heat capacity in the power - law expansion model decreases very rapidly. Of course, the mechanism for this suppression must be understood from the thermodynamic analogy due to systems with imaginary temperatures and those with negative heat capacities occur in different contexts [4,14–16], but their thermodynamic behavior has a common physical basis. The imaginary nature of β can be a consequence of the quantum nature of the matter field fluctuations in the infrared's environment (i.e., of the ultraviolet sector). The temporal dependence for β and C_V in the power-law expansion is due to the interaction of the inflaton field, which manifests itself in the equation of motion (4) for the quantum fluctuations through the time-dependent mass parameter $\mu(t) \equiv k_0/a \sim t^{-1}$. This effect generates — for a power - law expanding universe — a superexponential increasing for the number of degrees of freedom (i.e., the number of states) in the infrared sector, rather a exponential increasing of states founded in a de Sitter expanding universe due to β and C_V remain constant. . So, in a power-law expanding universe the production of new states is so copious that an inflow of energy actually drives the background temperature $(\beta\beta^*)^{-1/2}$ down asymptotically to zero, meanwhile in a de Sitter expansion the inflow of energy go entirely into producing new states, leaving the background temperature constant.

Finally, super Hubble matter field fluctuations with negative heat capacity during inflation describes exponential or superexponential growth of the number of states, which is a characteristic of nonequilibrium thermodynamical systems. It shows that a more dynamical approach to the statistical mechanics of inflation might be necessary.

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